Int. J. Heat Mass Transfer. Vol. 29, No. 3, pp. 381–390, 1986
Printed in Great Britain

0017-9310/86 $3.00+0.00

A thermodynamical modelling of fluid flows through porous media: application to natural convection

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(*Received* 30 *April* 1985 *and in final form* 18 *September* 1985)

Abstract—Extended irreversible thermodynamics is employed to model porous media. The basic ideas are to raise the heat flux and the diffusion flux to the status of independent variables and to view the porous medium as a binary mixture formed by a perfectly rigid solid and a fluid. Among other results, Brinkman's law receives a sound thermodynamic justification and the range of applicability of Darcy's law is precisely stated. The model is applied to study natural convection in a thin porous layer heated from below (Bénard's instability). The problem is treated by a linear perturbation analysis: the role of the boundary conditions and the permeability on the onset of convection is stressed; special attention is also drawn on the effects of a temperature-dependent viscosity.

1. INTRODUCTION

THE AIM of this paper is to present a coherent mathematical modelling of fluid flows through porous media. This is achieved within the framework of extended irreversible thermodynamics [1–4]. In this formalism, the dissipative fluxes, like the heat flux and the diffusion fluxes, are considered as independent variables besides the classical ones, formed by the density, velocity, temperature and concentrations fields. The time evolution of the extra variables is assumed to be governed by first-order time differential equations.

In the study of porous materials, the basic relation is the law of Darcy [5]; it expresses that for an isothermal fluid, moving with a slow steady velocity $\mathbf{v}$ through a porous, homogeneous and isotropic bed under the action of a pressure gradient ∇p, one has

$$\nabla p + \frac{\eta}{K}\mathbf{v} = 0, \tag{1}$$

η is the dynamic viscosity of the fluid, K the permeability of the medium.

But it was soon noticed that Darcy's law was inconsistent with the no-slip condition. This motivated Brinkman [6] to modify Darcy's equation by adding a second-order velocity gradient resulting in

$$\nabla p + \frac{\eta}{K}\mathbf{v} - \tilde{\eta}\nabla^2\mathbf{v} = 0, \tag{2}$$

where $\tilde{\eta}$ is the so-called effective viscosity.

Several interpretations [7–14] have been proposed to put the laws of Darcy and Brinkman on a sound basis. Most of them are formulated by using either the classical [15] or the rational [16] thermodynamics of irreversible processes. In the present study, porous media are approached from a different point of view, namely extended irreversible thermodynamics. It is seen that this scheme provides a simple derivation of the Darcy and Brinkman laws whose range of validity and applicability will be clearly stated.

One considers a porous medium consisting of a rigid body permeated by a one-component incompressible viscous fluid. It is assumed that the pore configuration has a scale length which is small compared to a macroscopic reference scale. The material is thus fine grained and may be viewed as homogeneous at the macroscopic level. Within these hypotheses, one is allowed to model the porous medium by means of a solid–fluid mixture [4, 7, 8] wherein every point in space is occupied simultaneously by rigid solid and fluid particles. Phase changes, chemical reactions as well as interfacial effects are ruled out.

In Section 2, the notation is introduced; the balance and constitutive equations are formulated. Limitations on their forms are imposed by the second law of thermodynamics and the principle of objectivity; these constraints are given explicitly in Section 3. A linearized version is proposed in Section 4; it is seen that it contains the Brinkman and Darcy models as particular cases. As illustration of our formalism, Bénard's convective instability is treated in Section 5. The prominent parts played by the permeability and a temperature-dependent viscosity are displayed. Final comments are found in Section 6.

2. NOTATION AND BASIC EQUATIONS

Subscripts s and f refer to the solid and the fluid parts of the medium, respectively.

The porosity is defined as the volume occupied by the fluid divided by the total volume, it can be expressed as

$$\varepsilon = \frac{\rho_f}{\gamma_f} \quad \varepsilon \in [0, 1], \tag{3}$$

where ρ_f represents the ratio of the mass of fluid and the

NOMENCLATURE

a_1, a_2	coupling coefficients
c_ε	specific heat
c_f, c_s	concentrations
d	thickness of the layer
D_1, D_2	diffusion coefficients
$\mathbf{F}$	specific body force
$\mathbf{I}$	identity tensor
$\mathbf{J}$	diffusion flux
K	permeability
$\mathscr{K}$	dimensionless permeability
p	pressure
Pr	Prandtl number
$\mathbf{q}$	heat flux
r	specific energy supply
Ra	Rayleigh number
s	specific entropy
t	time
T	temperature
u	specific internal energy
$\mathbf{v}$	velocity field
$\mathbf{V}$	velocity gradient tensor
w	normal component of the velocity
W	amplitude of the vertical velocity component
$\mathbf{x}$	space coordinate.

Greek symbols

α	heat expansion coefficient
γ	true density
∇	nabla operator
ε	porosity
η	dynamic viscosity
$\tilde{\eta}$	effective viscosity
θ	temperature
Θ	amplitude of the temperature field
κ	heat diffusivity
λ	heat conductivity
λ_1, λ_2	coupling coefficients
Λ_1, Λ_2	coupling coefficients
μ	chemical potential
ν	kinematic viscosity
ρ	density
σ	time growth constant
$\boldsymbol{\sigma}$	stress tensor
$\boldsymbol{\sigma}^v$	viscous stress tensor
τ_i	relaxation times ($i = 1, 2, 3$)
ψ	Helmholtz free energy.

Subscripts

f	refers to the fluid component
s	refers to the solid component.

Superscripts

·	material time derivative
*	dimensionless quantity
0	reference quantity
b	basic unperturbed solution
p	perturbed solution
T	transposition.

volume of the whole medium; γ_f, often referred to as the true density, is the ratio of the mass of fluid and the volume occupied by the fluid, γ_f is a constant for an incompressible fluid. For a saturated porous medium, one has, in terms of the solid phase,

$$1-\varepsilon = \frac{\rho_s}{\gamma_s}. \tag{4}$$

The total density ρ and the barycentric velocity $\mathbf{v}$ are defined by

$$\rho = \rho_f + \rho_s = \varepsilon\gamma_f + (1-\varepsilon)\gamma_s, \tag{5}$$

and

$$\rho\mathbf{v} = \varepsilon\gamma_f\mathbf{v}_f + (1-\varepsilon)\gamma_s\mathbf{v}_s, \tag{6}$$

where $\mathbf{v}_f$ and $\mathbf{v}_s$ denote the velocity of the fluid and the solid components, respectively.

The fluid and solid diffusion fluxes are given by

$$\mathbf{J}_f = \rho_f(\mathbf{v}_f - \mathbf{v}), \quad \mathbf{J}_s = \rho_s(\mathbf{v}_s - \mathbf{v}), \tag{7}$$

with, as a consequence of (6),

$$\mathbf{J}_f + \mathbf{J}_s = 0. \tag{8}$$

For further purposes, we introduce also the concentrations c_f and c_s:

$$c_f = \frac{\rho_f}{\rho} = \varepsilon\frac{\gamma_f}{\rho}, \quad c_s = \frac{\rho_s}{\rho} = \frac{(1-\varepsilon)\gamma_s}{\rho}, \tag{9}$$

with

$$c_f + c_s = 1. \tag{10}$$

The equations of evolution are the usual balance laws of densities, total momentum and energy:

$$\dot{\rho}_\alpha = -\rho_\alpha\nabla\cdot\mathbf{v} - \nabla\cdot\mathbf{J}_\alpha \quad \text{or} \quad \rho\dot{c}_\alpha = -\nabla\cdot\mathbf{J}_\alpha, \tag{11a, b}$$

$$\rho\dot{\mathbf{v}} = \nabla\cdot\boldsymbol{\sigma} + \sum_\alpha \rho_\alpha\mathbf{F}_\alpha, \tag{12}$$

$$\rho\dot{u} = -\nabla\cdot\mathbf{q} + \boldsymbol{\sigma}:\mathbf{V} + \sum_\alpha \mathbf{J}_\alpha\cdot\mathbf{F}_\alpha + \rho r. \tag{13}$$

α stands for f and s, respectively, while a dot over a character denotes the material time derivative, $\boldsymbol{\sigma}$ is a symmetric stress tensor, conveniently decomposed according to

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}^v, \tag{14}$$

where p is the thermodynamic equilibrium pressure, $\mathbf{I}$ the identity tensor, $\boldsymbol{\sigma}^v$ the viscous stress tensor. The remaining undefined quantities in (11)–(13) are: u, the specific internal energy; $\mathbf{q}$, the heat flux vector; r, the specific energy supply; $\mathbf{F}_\alpha$, the specific body force, assumed to be identical for both constituents; $\mathbf{V}$, the symmetric velocity gradient tensor

$$\mathbf{V} = \tfrac{1}{2}[\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}]; \tag{15}$$

in (15), superscript T means transposition. We also introduce a positive empirical temperature θ. Note that one has avoided using momentum and energy equations for the individual constituents because such relations involve undefined quantities—partial stress tensors, partial momentum supply, partial heat fluxes, etc. After summation of (11) on the two constituents, one recovers the law of conservation of the total mass:

$$\dot{\rho} = -\rho\nabla\cdot\mathbf{v}. \tag{16}$$

We now formulate the problem in the framework of extended irreversible thermodymanics (for a review, see [3]). Our final objective is to determine the behaviour of the seven independent variables ρ_{f}, ρ_{s} (or ρ and c_{f}), $\mathbf{v}$, θ, $\mathbf{q}$, $\boldsymbol{\sigma}^{\mathrm{v}}$, $\mathbf{J}_{\mathrm{f}}$ (or $\mathbf{J}_{\mathrm{s}}$) as a function of space $\mathbf{x}$ and time t. In contrast with earlier theories of non-equilibrium thermodynamics, the fluxes $\mathbf{q}$, $\boldsymbol{\sigma}^{\mathrm{v}}$ and $\mathbf{J}_{\mathrm{f}}$ are no longer considered as dependent variables but instead they are treated as independent quantities, at the same level as the classical variables ρ_{f}, ρ_{s}, $\mathbf{v}$ and θ.

To compensate for the lack of evolution equations, supplementary rate equations for the dissipative fluxes are introduced besides the classical balance equations of mass, momentum and energy; they are assumed to be first-order differential equations of the form

$$\tau_1\dot{\mathbf{q}} = \hat{\mathbf{q}}(C), \tag{17}$$

$$\tau_2\dot{\mathbf{J}}_{\mathrm{f}} = \hat{\mathbf{J}}(C), \tag{18}$$

$$\tau_3\dot{\boldsymbol{\sigma}}^{\mathrm{v}} = \hat{\boldsymbol{\sigma}}(C), \tag{19}$$

τ_1, τ_2 and τ_3 represent relaxation times; $\hat{\mathbf{q}}$, $\hat{\boldsymbol{\sigma}}$ and $\hat{\mathbf{J}}$ are response functions supposed to be functions of the whole set C of variables; C stands for

$$C = \rho,\ c_{\mathrm{f}},\ \theta,\ \nabla\rho,\ \nabla c_{\mathrm{f}},\ \nabla\theta,\ \mathbf{V},\ \mathbf{q},\ \boldsymbol{\sigma}^{\mathrm{v}},\ \mathbf{J}_{\mathrm{f}}. \tag{20}$$

In (20), the velocity field $\mathbf{v}$ has been replaced by the velocity gradient tensor $\mathbf{V}$ to be in agreement with the principle of objectivity [17]. The latter demands also that the response functions are isotropic functions of the variables with respect to Euclidean transformations and that the non-objective material time-derivatives be replaced by objective ones [3]. However, since we are primarily concerned with the formulation of a linear theory, the lack of objectivity of the time-derivatives is not important within the present context. The gradients of the 'slow' variables ρ, c_{f} and θ have been introduced among the set (20) of variables in order to include the effects resulting from density and temperature inhomogeneities. As stated earlier, the deformation of the solid constituent is neglected; such an effect could be formally included in the present formalism but it would obscure the specific points that are intended to be enlightened.

We are faced with a system of seven evolution equations (11)–(13) and (17)–(19) involving only one single unknown quantity, the internal energy u. The problem is well posed by expressing u in terms of C, via the constitutive relation

$$u = \hat{u}(C). \tag{21}$$

Limitations on the response functions $\hat{\mathbf{q}}$, $\hat{\boldsymbol{\sigma}}$, $\hat{\mathbf{J}}$ and $\hat{u}$ are introduced by the second law of thermodynamics, which, in its general form is given by [3, 16]:

$$\rho\dot{s} \geqslant -\nabla\cdot\frac{1}{\theta}(\mathbf{q}-\mu\mathbf{J}_{\mathrm{f}})+\rho\frac{r}{\theta}; \tag{22}$$

s is the specific entropy, r/θ the specific entropy supply, and $\mu(=\mu_{\mathrm{f}}-\mu_{\mathrm{S}})$ the chemical potential.

The entropy inequality (22) introduces one new unknown s, which implies that a supplementary constitutive equation is needed:

$$s = \hat{s}(C). \tag{23}$$

Moreover, since the temperature has been selected as independent variable, it is more natural to use the Helmholtz free energy $\psi = u-\theta s$ rather than the internal energy as dependent variable. It follows that the constitutive relation (21) must be replaced by

$$\psi = \hat{\psi}(C). \tag{24}$$

3. LIMITATIONS IMPOSED ON THE RESPONSE FUNCTIONS

In this section, we shall not enter into the details of calculations which have been explicitely developed in the past [2–4].

Elimination of r between the energy balance (13) and the entropy inequality (22) yields

$$-\rho(\dot{\psi}+s\dot{\theta})-\frac{1}{\theta}\nabla\theta\cdot\mathbf{q}+\boldsymbol{\sigma}:\mathbf{V}-\theta\nabla\cdot\left(\frac{\mu}{\theta}\mathbf{J}_{\mathrm{f}}\right) \geqslant 0. \tag{25}$$

To obtain the restrictions on the various response functions, one must differentiate (24) with respect to the time and substitute it into (25).

This operation results in

$$\begin{aligned}&-\rho\left(\frac{\partial\hat{\psi}}{\partial\theta}+\hat{s}\right)\dot{\theta}+\left(\frac{\partial\hat{\psi}}{\partial c_{\mathrm{f}}}-\mu\right)\nabla\cdot\mathbf{J}_{\mathrm{f}}+\left(\rho^2\frac{\partial\hat{\psi}}{\partial\rho}-p\right)\nabla\cdot\mathbf{v}\\&-\rho\left(\frac{\partial\hat{\psi}}{\partial\mathbf{q}}\cdot\dot{\mathbf{q}}+\frac{\partial\hat{\psi}}{\partial\mathbf{J}_{\mathrm{f}}}\cdot\dot{\mathbf{J}}_{\mathrm{f}}+\frac{\partial\hat{\psi}}{\partial\boldsymbol{\sigma}^{\mathrm{v}}}:\dot{\boldsymbol{\sigma}}^{\mathrm{v}}\right)-\rho\frac{\partial\hat{\psi}}{\partial(\nabla\theta)}\cdot\dot{\overline{\nabla\theta}}\\&-\rho\frac{\partial\hat{\psi}}{\partial(\nabla\rho)}\cdot\dot{\overline{\nabla\rho}}-\rho\frac{\partial\hat{\psi}}{\partial(\nabla c_{\mathrm{f}})}\cdot\dot{\overline{\nabla c_{\mathrm{f}}}}\\&-\frac{\partial\psi}{\partial\mathbf{V}}:\dot{\mathbf{V}}-\frac{1}{\theta}\nabla\theta\cdot\mathbf{q}+\boldsymbol{\sigma}^{\mathrm{v}}:\mathbf{V}-\theta\mathbf{J}_{\mathrm{f}}\cdot\nabla\left(\frac{\mu}{\theta}\right)\geqslant 0,\end{aligned} \tag{26}$$

where use has been made of (11b) and (16) to eliminate $\dot{c}_{\mathrm{f}}$ and $\dot{\rho}$; the decomposition (14) has also been employed. It is noticed that the inequality (26) is linear in the quantities $\dot{\theta}$, $\nabla\cdot\mathbf{J}_{\mathrm{f}}$, $\dot{\overline{\nabla\theta}}$, $\dot{\overline{\nabla\rho}}$, $\dot{\overline{\nabla c_{\mathrm{f}}}}$ and $\dot{\mathbf{V}}$. Clearly, the positiveness of (26) is assured at the condition that the factors of these quantities vanish. This requirement leads to the following relations:

$$\frac{\partial\hat{\psi}}{\partial\theta}+\hat{s}=0, \quad \mu=\frac{\partial\hat{\psi}}{\partial c_{\mathrm{f}}}, \tag{27}$$

$$\frac{\partial\hat{\psi}}{\partial(\nabla\rho)}=\frac{\partial\hat{\psi}}{\partial(\nabla c_{\mathrm{f}})}=\frac{\partial\hat{\psi}}{\partial(\nabla\theta)}=\frac{\partial\hat{\psi}}{\partial\mathbf{V}}=0. \tag{28}$$

The results (27) are well-known while (28) expresses that ψ (as well as s) is independent of the gradients of the slow variables; as a consequence ψ and s can be written as

$$\psi=\hat{\psi}(\rho, c_{\mathrm{f}}, \theta, \mathbf{q}, \boldsymbol{\sigma}^{\mathrm{v}}, \mathbf{J}_{\mathrm{f}}), \tag{29}$$

$$s=\hat{s}(\rho, c_{\mathrm{f}}, \theta, \mathbf{q}, \boldsymbol{\sigma}^{\mathrm{v}}, \mathbf{J}_{\mathrm{f}}). \tag{30}$$

With the above results in mind, the entropy inequality reads as

$$-\rho\left(\frac{\partial\hat{\psi}}{\partial\mathbf{q}}\cdot\frac{1}{\tau_1}\hat{\mathbf{q}}+\frac{\partial\hat{\psi}}{\partial\mathbf{J}_{\mathrm{f}}}\cdot\frac{1}{\tau_2}\hat{\mathbf{J}}_{\mathrm{f}}+\frac{\partial\hat{\psi}}{\partial\boldsymbol{\sigma}^{\mathrm{v}}}:\frac{1}{\tau_3}\hat{\boldsymbol{\sigma}}\right)$$
$$+\left(\rho^2\frac{\partial\hat{\psi}}{\partial\rho}-p\right)\nabla\cdot\mathbf{v}$$
$$-\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta+\boldsymbol{\sigma}^{\mathrm{v}}:\mathbf{V}-\theta\mathbf{J}_{\mathrm{f}}\nabla\left(\frac{\mu}{\theta}\right)\geqslant 0. \tag{31}$$

Set

$$\bar{p}=\rho^2\frac{\partial\hat{\psi}}{\partial\rho}, \tag{32}$$

where $\bar{p}$ is a generalized pressure which reduces to the equilibrium pressure p at equilibrium. Collection of the results (27), (28) and (32) leads to the generalized Gibbs equation

$$\mathrm{d}\psi=-s\,\mathrm{d}\theta+\frac{\bar{p}}{\rho^2}\,\mathrm{d}\rho+\mu\,\mathrm{d}c_{\mathrm{f}}+\frac{\partial\psi}{\partial\mathbf{q}}\cdot\mathrm{d}\mathbf{q}$$
$$+\frac{\partial\psi}{\partial\mathbf{J}_{\mathrm{f}}}\cdot\mathrm{d}\mathbf{J}_{\mathrm{f}}+\frac{\partial\psi}{\partial\boldsymbol{\sigma}^{\mathrm{v}}}:\mathrm{d}\boldsymbol{\sigma}^{\mathrm{v}}. \tag{33}$$

In the classical theory of irreversible processes [15], only the first three terms in the RHS are present.

4. RESULTS NEAR EQUILIBRIUM: THE BRINKMAN AND DARCY LAWS

By equilibrium we mean a state with time-independent variables, neither temperature nor velocity gradients and absence of dissipative fluxes. Near equilibrium, it is reasonable to approximate the evolution equations by their linearized forms

$$\tau_1\,\partial_t\mathbf{q}=-\mathbf{q}-\lambda\nabla\theta-a_1\mathbf{J}_{\mathrm{f}}-\Lambda_1\nabla\rho-D_1\nabla c_{\mathrm{f}}, \tag{34}$$

$$\tau_2\,\partial_t\mathbf{J}_{\mathrm{f}}=-\mathbf{J}_{\mathrm{f}}-D_2\nabla c_{\mathrm{f}}-a_2\mathbf{q}-\Lambda_2\nabla\rho-\lambda_2\nabla\theta, \tag{35}$$

$$\tau_3\,\partial_t\boldsymbol{\sigma}^{\mathrm{v}}=-\boldsymbol{\sigma}^{\mathrm{v}}+\eta\mathbf{V}, \tag{36}$$

where the various coefficients may depend on ρ, c_{f} and θ. For steady processes and no coupling, one recovers the usual Fourier, Fick and Newton–Navier equations. It is therefore justified to identify the coefficients λ, D_2 and η respectively with the heat conductivity, the diffusion coefficient and the shear viscosity; the coupling coefficients D_1, a_1, Λ_1, a_2, Λ_2 and λ_2 describe the interactions between the heat and mass flows. Equation (34) generalizes the Vernotte–Cattaneo [18, 19] relation

$$\tau_1\,\partial_t\mathbf{q}=-\mathbf{q}-\lambda\nabla\theta, \tag{37}$$

while (36) is nothing but the Maxwell relation [20]. If one replaces in (35) the flux $\mathbf{J}_{\mathrm{f}}$ by its definition (7) and makes use of the momentum balance (12), one obtains a balance equation for the fluid velocity given by

$$\tau_2\rho_{\mathrm{f}}\,\partial_t\mathbf{v}_{\mathrm{f}}=-\rho_{\mathrm{f}}(\mathbf{v}_{\mathrm{f}}-\mathbf{v})+\frac{\rho_{\mathrm{f}}}{\rho}\tau_2\left(\nabla\cdot\boldsymbol{\sigma}+\sum\rho_\alpha\mathbf{F}_\alpha\right)$$
$$-a_2\mathbf{q}-\lambda_2\nabla\theta-\Lambda_2\nabla\rho-D_2\nabla c_{\mathrm{f}}. \tag{38}$$

Recalling that the solid is rigid ($\mathbf{v}_{\mathrm{s}}=0$), one has

$$\mathbf{v}=\frac{\rho_{\mathrm{f}}}{\rho}\mathbf{v}_{\mathrm{f}}. \tag{39}$$

After substitution of (39) in (38) and elimination of $\mathbf{q}$ by means of (34) wherein τ_1 is assumed small compared to τ_2, one obtains

$$\tau_2\rho_{\mathrm{f}}\,\partial_t\mathbf{v}_{\mathrm{f}}=-\frac{\rho_{\mathrm{f}}\rho_{\mathrm{s}}}{\rho}(1-a_1a_2)\mathbf{v}_{\mathrm{f}}$$
$$+\frac{\rho_{\mathrm{f}}}{\rho}\tau_2\left(-\nabla p+\eta\nabla^2\mathbf{v}+\sum\rho_\alpha\mathbf{F}_\alpha\right)+\nabla\theta(\lambda a_2-\lambda_2)$$
$$+(a_2D_1-D_2)\nabla c_{\mathrm{f}}+(a_2\Lambda_1-\Lambda_2)\nabla\rho. \tag{40}$$

This equation is the key relation of the present note. In contrast to other works concerned with porous media, it is the acceleration rather than the velocity which is expressed in terms of the set of independent variables. The consequences on a typical problem of fluid mechanics, like Rayleigh–Bénard thermal convection, will be examined in the next section.

For further comparison, let us define the permeability K by the relation

$$\frac{\rho_{\mathrm{s}}(1-a_1a_2)}{\tau_2}=\frac{\eta}{K}. \tag{41}$$

Since the coupling coefficients a_1 and a_2 are generally small compared to unity, it is reasonable to write

$$K=\tau_2\frac{\eta}{\rho_{\mathrm{s}}}=\tau_2\frac{\eta}{(1-\varepsilon)\gamma_{\mathrm{s}}}. \tag{42}$$

According to the kinetic theories, e.g. [22, 23], the relaxation time τ_2 is related to the mean free time or the reciprocal of the collision frequency of diffusing particules; for usual fluids, τ_2 takes values ranging between 10^{-8} and 10^{-13} s. The result (42) indicates that K is of the same order of magnitude, i.e. $K\sim 10^{-8}$–10^{-13} m^2.

Within the limit of an incompressible and isothermal fluid for which $\nabla\rho=\nabla c_{\mathrm{f}}=\nabla\theta=0$, the balance equation (40) simply becomes

$$\rho\,\partial_t\mathbf{v}_{\mathrm{f}}=-\frac{\eta}{K}\mathbf{v}_{\mathrm{f}}-\nabla p+\frac{\rho_{\mathrm{f}}}{\rho}\eta\nabla^2\mathbf{v}_{\mathrm{f}}+\sum\rho_\alpha\mathbf{F}_\alpha. \tag{43}$$

For steady flows and in the absence of external forces, one recovers Brinkman's relation

$$\nabla p + \frac{\eta}{K}\mathbf{v}_{\mathrm{f}} - \tilde{\eta}\nabla^2\mathbf{v}_{\mathrm{f}} = 0, \tag{44}$$

where $\tilde{\eta} = (\rho_{\mathrm{f}}/\rho)\eta$ is the effective viscosity. Clearly (43) reduces to Navier–Stokes' equation as $K \to \infty$ and to Darcy's relation

$$\nabla p + \frac{\eta}{K}\mathbf{v}_{\mathrm{f}} = 0, \tag{45}$$

if $K/d^2 \ll 1$ (d denotes a spatial length scale).

Darcy's law is usually selected when the volume of the solid particules is larger than the volume occupied by the fluid, otherwise Brinkman's law is preferred.

It is important to realize that Brinkman's law has been established on sound macroscopic thermodynamical bases and appears as a simplified form of the evolution equation for the fluid flow.

Our next objective is to formulate the energy balance equation. Using the definition

$$u = \psi + \theta s$$

and the Gibbs relation (33), one obtains within the linear approximation:

$$\partial_t u = c_{\mathrm{e}}\,\partial_t\theta - \left[\theta^2\frac{\partial}{\partial\theta}\left(\frac{p}{\rho^2\theta}\right)\right]_{\mathrm{e}}\partial_t\rho - \left[\theta^2\frac{\partial}{\partial\theta}\left(\frac{\mu}{\theta}\right)\right]_{\mathrm{e}}\partial_t c_{\mathrm{f}} + O(2), \tag{46}$$

where subscript e refers to a quantity measured at equilibrium, parameter c_{e} is the heat capacity evaluated at equilibrium. Substitution of (46) in (13) yields the equation governing the temperature field:

$$\rho c_{\mathrm{e}}\,\partial_t\theta = -\nabla\cdot\mathbf{q} - \left[p + \frac{\theta}{\rho}\frac{\partial}{\partial\theta}\left(\frac{p}{\rho^2\theta}\right)\right]_{\mathrm{e}}\nabla\cdot\mathbf{v} - \left[\theta^2\frac{\partial}{\partial\theta}\left(\frac{\mu}{\theta}\right)\right]_{\mathrm{e}}\nabla\cdot\mathbf{J}_{\mathrm{f}} + \boldsymbol{\sigma}:\mathbf{V} + \rho r. \tag{47}$$

Although the model developed in this section is rather simple, there is no fundamental difficulty to extend it by including, for instance, the deformation of the solid matrix and the effects of interface between the solid and liquid phases. However the objective of the present note is not to propose a complete theory of porous media but rather to emphasize the consequences resulting from a descriptive within the framework of extended irreversible thermodynamics.

5. NATURAL CONVECTION IN POROUS MEDIA (BENARD PROBLEM)

The model proposed in Section 4 is used to describe natural convection in a horizontal layer of a saturated porous medium heated from below. The layer, of thickness d, extends to infinity; it is bounded above and below by either rigid or stress-free plane surfaces, moreover the boundaries are either perfectly heat conducting or adiabatically insulated. The porous medium is saturated by an incompressible fluid of dynamic viscosity η. It is assumed that the heat flux and the viscous stress relax mush faster than the diffusion flux ($\tau_1 \ll \tau_2$, $\tau_3 \ll \tau_2$), which is expected to be the dominant effect in the present problem. Throughout this section, only infinitesimally small disturbances are considered.

In the unperturbed reference state, the porous medium is at rest with a temperature drop ΔT between the lower and upper bounding surfaces. Within Boussinesq approximation, the relevant equations are

$$\nabla\cdot\mathbf{v}_{\mathrm{f}} = 0, \tag{48}$$

$$\nabla\cdot\mathbf{J}_{\mathrm{f}} = 0, \tag{49}$$

$$\partial_t\mathbf{v}_{\mathrm{f}} = -\frac{\rho_{\mathrm{s}}^{\circ}}{\rho^{\circ}\tau_2}\mathbf{v}_{\mathrm{f}} + \frac{1}{\rho^{\circ}}\left(-\nabla p + \frac{\rho_{\mathrm{f}}^{\circ}}{\rho^{\circ}}\eta\nabla^2\mathbf{v}_{\mathrm{f}}\right) + \frac{\rho}{\rho^{\circ}}\mathbf{g} - \frac{1}{\rho_{\mathrm{f}}^{\circ}\tau_2}(a_2\mathbf{q} + \lambda_2\nabla\theta), \tag{50}$$

$$\rho^{\circ}c_{\mathrm{e}}\,\partial_t\theta = -\nabla\cdot\mathbf{q}, \tag{51}$$

$$\mathbf{q} = -\mathrm{a}_1\frac{\rho_{\mathrm{f}}^{\circ}\rho_{\mathrm{s}}^{\circ}}{\rho^{\circ}}\mathbf{v}_{\mathrm{f}} - \lambda\nabla\theta, \tag{52}$$

the density ρ is given by the state equation

$$\rho = \rho^{\circ}[1 - \alpha(\theta - \theta^{\circ})],$$

α is the heat expansion coefficient and superscript $^{\circ}$ refers to a constant value calculated at the reference temperature θ°, say the steady temperature of the lower face of the layer. Equation (48) indicates that the fluid is incompressible and (49) that the concentration remains constant, the latter being a consequence of the former. Equation (50) is a reformulation of (38) wherein Boussinesq hypothesis has been introduced. Within the same approximation, the energy balance takes the simple form (51) (no viscous dissipation) while (52) is a simplified form of (34) wherein all the above mentioned hypotheses have been used, (52) may also be viewed as an extension of the Fourier law involving an extra linear term in the fluid velocity.

From now on the procedure is routine. One determines firstly the solution in the basic quiescent state given by

$$\mathbf{v}_{\mathrm{f}}^{\mathrm{b}} = 0, \quad \theta^{\mathrm{b}} = -\frac{\Delta T}{d}z + \theta_0, \quad \mathbf{q}^{\mathrm{b}} = \lambda\frac{\Delta T}{d}\mathbf{e}_z, \tag{53}$$

where $\mathbf{e}_z$ is the unit vector pointing upwards and index b refers to the basic solution. After onset of convection, the variables take the values θ, $\mathbf{v}_{\mathrm{f}}$ and $\mathbf{q}$ and one defines perturbed quantities by

$$\theta^{\mathrm{p}} = \theta - \theta^{\mathrm{b}}, \quad \mathbf{v}_{\mathrm{f}}^{\mathrm{p}} = \mathbf{v}_{\mathrm{f}}, \quad \mathbf{q}^{\mathrm{p}} = \mathbf{q} - \mathbf{q}^{\mathrm{b}}, \quad p^{\mathrm{p}} = p - p^{\mathrm{b}}. \tag{54}$$

Eliminating $\mathbf{q}^{\mathrm{p}}$ with the help of (52), equations (51) and (50) reduce to

$$\partial_t\theta^{\mathrm{p}} - \frac{\Delta T}{d}w = \kappa\nabla^2\theta^{\mathrm{p}}, \tag{55}$$

$$\partial_t \mathbf{v}_f = -\frac{\nu}{K}\mathbf{v}_f - \frac{1}{\rho}\nabla p^p + \frac{\rho_f}{\rho}\nu\nabla^2\mathbf{v}_f + \alpha\theta^p\mathbf{g} + \frac{1}{\rho_f\tau_2}(\lambda a_2 - \lambda_2)\nabla\theta^p, \quad (56)$$

superscripts zero affecting the densities have been dropped, κ is the heat diffusivity, ν the kinematic viscosity, w the vertical component of the fluid velocity. Elimination of ∇p^p and $\nabla\theta^p$ in (56) and adimensionalization of z, t, w, θ by means of d, d^2/κ, κ/d and ΔT yields the following set of equations:

$$\partial_t\theta^* - w^* = \nabla^2\theta^*, \quad (57)$$

$$\frac{1}{Pr}\partial_t\nabla^2 w^* = Ra\,\nabla_1^2\theta^* - \frac{1}{\mathscr{K}}\nabla^2 w^* + \nabla^4 w^*, \quad (\nabla_1^2 = \partial_{xx}^2 + \partial_{yy}^2). \quad (58)$$

In (57) and (58), an asterisk denotes a dimensionless field; Pr, Ra and $\mathscr{K}$ are the Prandtl, Rayleigh and permeability dimensionless numbers, respectively defined by

$$Pr = \frac{\rho_f}{\rho}\frac{\nu}{\kappa}, \quad Ra = \frac{\alpha q \Delta T d^3}{(\rho_f/\rho)\nu\kappa}, \quad \mathscr{K} = \frac{\rho_f K}{\rho d^2}. \quad (59)$$

Apart from the factor ρ_f/ρ, these expressions are classical. When $\mathscr{K} \to \infty$, one recovers the well-known relations governing the Bénard–Rayleigh problem. In the limit $\mathscr{K} \to 0$, (57) and (58) are identical to the equations derived from Darcy's law [24, 25].

We search for solutions of the form

$$(w^*, \theta^*) = [W(z), \Theta(z)]\exp(i\mathbf{k}\cdot\mathbf{x} + \sigma t) \quad (60)$$

wherein $W(z)$ and $\theta(z)$ are the amplitudes of the disturbed velocity and temperature fields, $\mathbf{k}$ the horizontal wavenumber, $\mathbf{x}$ the horizontal position vector, σ the time growth constant. At marginal stability, not only the real part (Re σ) but also the imaginary part (Im σ) vanish since the set (57)–(58) coupled with the appropriate boundary conditions is self-adjoint; as a consequence, exchange of stability holds. After substitution of (60) in (57) and (58), one obtains the following equations for the amplitudes at marginal stability:

$$(\mathrm{D}^2 - k^2)^2 W - \mathscr{K}^{-1}(\mathrm{D}^2 - k^2)W = \mathscr{K}^2 Ra\Theta, \quad (61)$$

$$(\mathrm{D}^2 - k^2)\Theta = -W, \quad (62)$$

where D stands for ∂_z.

The associated boundary conditions at $z = 0$ and $z = 1$ are:

$$\Theta = 0 \text{ (isothermal surface)},$$
$$\mathrm{D}\Theta = 0 \text{ (thermally insulated surface)}, \quad (63)$$

$$W = \mathrm{D}W = 0 \text{ (rigid surface)}, \quad (64)$$

$$W = \mathrm{D}^2 W = 0 \text{ (free surface)}. \quad (65)$$

Elimination of Θ between (61) and (62) leads to

$$(\mathrm{D}^2 - k^2)^3 W - \mathscr{K}^{-1}(\mathrm{D}^2 - k^2)^2 W = -k^2 Ra W, \quad (66)$$

whose solution for two free surfaces, perfectly heat conducting, is

$$W = A\sin n\pi z \quad (n = 1, 2, \ldots). \quad (67)$$

The marginal stability curve, Ra vs k, is derived by replacing (67) in (66); one finds

$$Ra = \frac{(\pi^2 + k^2)^2}{k^2}[\mathscr{K}^{-1} + (\pi^2 + k^2)]. \quad (68)$$

The critical values R_c and k_c are obtained from $\partial Ra/\partial k = 0$ and given by

$$R_c = \frac{(\pi^2 + k_c^2)^2}{k_c^2}(A + k_c^2),$$

$$k_c^2 = \frac{-A + \sqrt{A^2 + 8\pi^2 A}}{4}, \quad (69)$$

where A stands for

$$A = \mathscr{K}^{-1} + \pi^2.$$

For $\mathscr{K} \to \infty$, one recovers

$$k_c = \frac{\sqrt{2}}{2}\pi, \quad R_c = \frac{27}{4}\pi^4, \quad (70)$$

and for $\mathscr{K} \to 0$,

$$k_c \to \pi, \quad \mathscr{K}R_c \to 4\pi^2, \quad (71)$$

$\mathscr{K}R_c$ is generally referred in the literature as the Rayleigh–Darcy number. The results (70) and (71) are well-known: (70) are the critical values obtained from the classical Navier–Stokes equations while (71) are the critical parameters corresponding to the use of Darcy's law, coupled with the boundary conditions $\Theta = W = 0$ [11]. Intermediate, critical values corresponding to $\mathscr{K}$ ranging from 10^{-9} to 10^{-2} are given in Table 1.

When both surfaces are not free and perfectly heat conducting, no analytical solution of (66) is available and numerical calculations are needed. Reported in Tables 2 and 3 are some critical values of R_c and k_c, as a function of $\mathscr{K}$, when the face below is rigid and isothermal while the face above is successively rigid, thermally conducting and free, thermally insulated. The problem has been solved by using a Galerkin technique whose details can be found in earlier works by the authors, e.g. [26–28].

Tables 1 and 2 indicate that, for two rigid and two free isothermal bounding surfaces, stability is increased with respect to Darcy's model. This is not surprising since the supplementary contribution introduced in our analysis is a viscous term whose effect is to inhibit

Table 1. Critical values $\mathscr{K}R_c$ and k_c: two free isothermal faces

$\mathscr{K}$	$\mathscr{K}R_c$	k_c
10^{-9}	39.48	3.14
10^{-6}	39.48	3.12
10^{-3}	40.25	3.11
10^{-2}	47.27	2.92
∞ (Navier–Stokes)	657.5 ($=R_c$)	2.22

Table 2. Critical values $\mathcal{K}R_c$ and k_c: two rigid isothermal faces

$\mathcal{K}$	$\mathcal{K}R_c$	k_c
0 (Darcy)	$4\pi^2$ (39.48)	π (3.14)
10^{-9}	41.5	3.22
10^{-6}	41.6	3.21
10^{-3}	43.7	3.20
∞ (Navier–Stokes)	1708 ($=R_c$)	3.12

instability. It is also noticed that the stability limit is practically unmodified when $\mathcal{K}$ varies from 0 to 10^{-3}. This confirms earlier conclusions by Katto and Masuoka [23] who stated that a linear stability analysis based on Darcy's law is acceptable even for large values of the permeability. Similarly, the critical wavenumber is only slightly affected by the variation of the permeability; one observes however a small decay for k_c, which means larger convection cells, when the permeability increases.

Most of the above observations can be repeated for Table 3. R_c as well as k_c are rather insensitive to the variations of $\mathcal{K}$; as expected, the system is less stable than in the case of two rigid boundaries. But unlike the results of Tables 1 and 2, one now notices a considerable difference with respect to Darcy's model: the critical Rayleigh–Darcy number differs by more or less 300% and the critical wave number by more than 150%.

The analysis of the results shows also that when the permeability tends to infinity, instability is not altered by the presence of the porous medium. The system behaves like an ordinary fluid, with critical wave Rayleigh numbers equal to 657.5, 669 and 1708 according to the boundary conditions. In contrast, for small values of the permeability, the Navier–Stokes equations must be discarded on behalf of more elaborate models including explicitly the effects of the porosity.

All the results presented up to now are based on the Boussinesq approximation stating, among other simplifications, that the viscosity of the fluid is independent of the temperature. In the following, we relax this constraint by assuming that the viscosity depends linearly on the temperature [29, 30]:

$$\nu = \nu^\circ[1+(\partial_T\nu)^\circ\Delta T]. \tag{72}$$

As a consequence, the term in $\nu\nabla^2\mathbf{v}_f$ must everywhere be substituted by $\nabla\cdot(\nu\nabla\mathbf{v}_f)$. It is readily checked that at marginal stability, the basic relation (66) is replaced by

$$(1-R_\nu z)[(\mathrm{D}^2-k^2)^3W-\mathcal{K}^{-1}(\mathrm{D}^2-k^2)^2W]$$
$$-R_\nu[4(\mathrm{D}^2-k^2)^2\mathrm{D}W-3\mathcal{K}^{-1}(\mathrm{D}^2-k^2)\mathrm{D}W]$$
$$=-k^2RaW, \tag{73}$$

where R_ν stands for the dimensionless viscosity number

$$R_\nu=\frac{(\partial_T\nu)^\circ}{\nu^\circ}\Delta T. \tag{74}$$

Table 3. Critical values $\mathcal{K}R_c$ and k_c: a lower rigid isothermal and an upper free insulated face

$\mathcal{K}$	$\mathcal{K}R_c$	k_c
0 (Darcy)	π^2 (9.87)	$\pi/2$ (1.57)
10^{-9}	27.3	2.30
10^{-6}	27.3	2.29
10^{-3}	28.5	2.28
∞ (Navier–Stokes)	669 ($=R_c$)	2.08

For common fluids, R_ν takes values ranging from -0.5 for oil to 0.5 for air [30]. The eigenvalue problem has been solved for two free and perfectly heat conducting surfaces in the Appendix. The critical Rayleigh–Darcy numbers are listed in Table 4 as a function of the dimensionless number R_ν, for three values of $\mathcal{K}$, namely $\mathcal{K}=0$, 10^{-2} and ∞.

The critical wavenumbers are not reported in Table 4 because they remain close to the values calculated for $R_\nu=0$. In contrast, the critical Rayleigh–Darcy numbers are appreciably affected by the temperature dependence of the viscosity. As a matter, $\mathcal{K}R_c$ varies by about 25% when going from $R_\nu=0$ to $R_\nu=-0.5$ and by more than 35% passing from $R_\nu=0$ to $R_\nu=0.5$. The stability limit is raised with increasing absolute values of R_ν for $R_\nu<0$ (i.e. in liquids) but for $R_\nu>0$ (i.e. in gases), the stability limit is lowered when R_ν is growing. It is important to note that the variation of the viscosity with the temperature plays a decisive role in determining the onset of stability in Bénard–Darcy problem.

6. FINAL COMMENTS

The present note has two objects in view: to describe a class of porous media within the framework of extended irreversible thermodynamics and to apply the proposed model to the Bénard–Darcy convection problem.

It is shown that extended irreversible thermodynamics is particularly well suited for modelling porous systems. Of course, in formulating the present model, several simplifications have been introduced: surface effects have been omitted, as well as changes of phase or deformations of the solid matrix. But our primary objective was not to give a complete

Table 4. Critical values of $\mathcal{K}R_c$ for various assigned values of R_ν and $\mathcal{K}$: two free isothermal faces

	$\mathcal{K}$		
R_ν	0	10^{-2}	∞ (R_c)
-0.5	49	58.34	817
-0.3	45.25	53.88	754.5
-0.1	41.45	49.32	690
0	39.48	46.99	657.5
0.1	37.5	44.62	624.5
0.3	33.35	39.73	556.5
0.5	28.96	34.57	485.5

description of porous media but rather to extract the main new features resulting from an analysis based on extended irreversible thermodynamics.

One of the main interests of this work is to ground the Brinkman model on sound thermodynamic foundations. It is seen that Brinkman's equations arise as a particular expression of the evolution equation of the fluid flow, when the gradients of the slow variables (density and temperature) are negligible. The permeability also receives an original interpretation in terms of the relaxation time of the fluid flux.

To show that the proposed model is not a mere abstraction, it is used to describe natural convection in a thin fluid layer heated from below (Rayleigh–Darcy problem). Differences with earlier treatments based on Darcy's law are emphasized. In particular, it is shown that the latter must be used with caution when the upper bounding surface is free: in the case of an upper thermally isolated boundary, the critical values of the Rayleigh–Darcy number may differ by a factor of three, according to whether our model or the Darcy model is selected. It is also shown under which circumstances, Darcy's model is reliable: in general, it provides reasonable results as long as the permeability is smaller than 10^{-3}. The effect of a temperature-dependent viscosity on the onset of convection can also be easily included into the frame of our formalism. The results indicate that this latter effect plays an important role: the critical Rayleigh–Darcy number can differ by as much as 35% in comparison to its value calculated for a constant viscosity.

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APPENDIX

Resolution of equation (73)

For the sake of clarity, let us recall expression (73):

$$(1-R_v z)\,[(\mathrm{D}^2-k^2)^3 W - K^{-1}(\mathrm{D}^2-k^2)^2 W] - R_v[4(\mathrm{D}^2-k^2)^2\mathrm{D}W - 3K^{-1}(\mathrm{D}^2-k^2)\mathrm{D}W] = -k^2 RaW. \quad \text{(A1)}$$

We seek solutions of the form

$$W = \sum_{n=0}^{\infty} c_n z^n, \quad \text{(A2)}$$

where the coefficients c_n are obtained by substituting (A2) in (A1); this operation yields the following set:

$$c_0, c_1, \ldots, c_5 \text{ are arbitrary,}$$

$$c_{n+6} = R_v c_{n+5}\frac{(n+5)}{(n+6)} + \frac{(3k^2+\mathscr{K}^{-1})}{(n+5)(n+6)}c_{n+4} - \frac{R_v[8k^2+3\mathscr{K}^{-1}+n(3k^2+\mathscr{K}^{-1})]c_{n+3}}{(n+4)(n+5)(n+6)}$$

$$-k^2\frac{(2\mathscr{K}^{-1}+3k^2)}{(n+3)(n+4)(n+5)(n+6)}c_{n+2}$$

$$+\frac{R_v k^2[4k^2+3\mathscr{K}^{-1}+n(3k^2+\mathscr{K}^{-1})]c_{n+1}}{(n+2)(n+3)(n+4)(n+5)(n+6)}$$

$$-\frac{k^2[Ra-k^2(k^2+\mathscr{K}^{-1})]}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}c_n$$

$$-\frac{R_v k^4(k^2+\mathscr{K}^{-1})c_{n+1}}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}. \tag{A3}$$

Without lack of generality, one is allowed to write W as the sum of six independent solutions

$$W=\sum_{n=0}^{5}c_n W_n \tag{A4}$$

such that

$$W_n = W \quad \text{for} \quad c_n = 1, \quad c_i = 0 \quad (i \neq n).$$

The c_n are derived from the boundary conditions; from

$$W(0) = \mathrm{D}^2W(0) = \Theta(0) = 0; \tag{A5}$$

it is readily seen that

$$c_0 = c_2 = 0, \tag{A6}$$

$$c_4 = R_v[12c_3-(\mathscr{K}^{-1}+2k^2)c_1]/24, \tag{A7}$$

while from

$$W(1) = \mathrm{D}^2W(1) = 0, \tag{A8}$$

one obtains:

$$c_5 = -(\alpha_1 c_1+\alpha_3 c_3), \tag{A9}$$

$$c_3 = -\frac{\beta_1}{\beta_3}c_1, \tag{A10}$$

$\alpha_1, \alpha_3, \beta_1$ and β_3 are respectively given by

$$\alpha_1 = \frac{W_1(1)-R_v[\mathscr{K}^{-1}+2k^2]W_4(1)/24}{W_5(1)}, \tag{A11}$$

$$\alpha_3 = [W_3(1)+R_vW_4(1)]/W_5(1), \tag{A12}$$

$$\beta_1 = \delta^2\alpha_1-\alpha_1\mathrm{D}^2W_5(1), \tag{A13}$$

$$\beta_3 = \delta^2\alpha_3(1)-\alpha_3\mathrm{D}^2W_5(1); \tag{A14}$$

where $\delta^i\alpha_1$ and $\delta^i\alpha_3$ ($i = 1,2,\ldots$) stand respectively for

$$\delta^i\alpha_1 = \mathrm{D}^iW_1(1)-R_v(\mathscr{K}^{-1}+2k^2)D^iW_4(1), \quad i=1,2,\ldots \tag{A15}$$

$$\delta^i\alpha_3 = \mathrm{D}^iW_3(1)+2R_vD^iW_4(1). \tag{A16}$$

Finally, applying the remaining boundary condition

$$\Theta(1) = 0 \tag{A17}$$

leads to a characteristic equation which can be solved directly for the eigenvalues Ra in terms of k, $\mathscr{K}$ and R_v; this equation takes the form

$$R_v(\mathscr{K}^{-1}+2k^2)\left\{\delta^1\alpha_1-\mathrm{D}W_5(1)\alpha_1-\frac{\beta_1}{\beta_3}[\delta^1\alpha_3-\alpha_3\mathrm{D}W_5(1)]\right\}$$
$$-2R_v\left\{\delta^3\alpha_1-\alpha_1\mathrm{D}^3W_5(1)-\frac{\beta_1}{\beta_3}\delta^3\alpha_3-\alpha_3\mathrm{D}^3W_5(1)\right\}$$
$$+(1-R_v)\left\{\delta^4\alpha_1-\alpha_1D^4W_5(1)-\frac{\beta_1}{\beta_3}[\delta^4\alpha_3-\alpha_3\mathrm{D}^4W_5(1)]\right\}=0. \tag{A18}$$

MODELISATION THERMODYNAMIQUE DES ECOULEMENTS DE FLUIDE A TRAVERS LES MILIEUX POREUX: APPLICATION A LA CONVECTION NATURELLE

Résumé—La thermodynamique des irréversibilités est employée pour modéliser les milieux poreux. L'idée de base est de considérer le flux thermique et le flux de diffusion comme des variables indépendantes et de considérer le milieu poreux comme un mélange binaire formé d'un solide rigide et d'un fluide. Parmi d'autres résultats, la loi de Brinkman reçoit une justification thermodynamique et le domaine d'application de la loi de Darcy est établi avec précision. Le modèle est appliqué à l'étude de la convection naturelle dans une couche poreuse mince chauffée par le bas (instabilité de Bénard). Le problème est traité par l'analyse linéaire de perturbation: le rôle des conditions aux limites et de la perméabilité sur l'apparition de la convection est dégagé: une attention spéciale est aussi portée sur les effets de la dépendance de la viscosité vis-à-vis de la température.

THERMODYNAMISCHES MODELL VON FLUID-STRÖMUNGEN IN PORÖSEN MEDIEN: ANWENDUNG AUF DIE NATÜRLICHE KONVEKTION

Zusammenfassung—Auf der Grundlage der irreversiblen Thermodynamik wird ein poröses Medium modelliert. Die Grundidee ist, den Wärme- und den Diffusionsstrom zu unabhängigen Variablen zu erheben und das poröse Medium als Zweistoffgemisch aus einem ideal starren Feststoff und einem Fluid zu betrachten. Neben anderen Ergebnissen erhält das Gesetz von Brinkman einen thermodynamischen Nachweis, und der Anwendungsbereich des Darcyschen Gesetzes wird genau aufgezeigt. Das Modell wird angewendet, um die natürliche Konvektion in einer von unten beheizten, dünnen, porösen Schicht zu untersuchen (Bénard Instabilität). Das Problem wird mit einem linearen Störungsansatz gelöst. Der Einfluß der Randbedingungen und der Permeabilität auf das Einsetzen der Konvektion wird dargestellt. Besonderes Augenmerk wird außerdem auf den Einfluß einer Temperaturabhängigkeit der Viskosität gerichtet.

ТЕРМОДИНАМИЧЕСКОЕ МОДЕЛИРОВАНИЕ ТЕЧЕНИЙ ЖИДКОСТИ ЧЕРЕЗ ПОРИСТЫЕ СРЕДЫ. СЛУЧАЙ СВОБОДНОЙ КОНВЕКЦИИ

Аннотация—Расширенная термодинамика необратимых процессов используется при моделировании пористых сред. Основные идеи состоят в том, чтобы анализировать тепловой и диффузионный потоки как независимые переменные и рассматривать пористую среду как бинарную смесь, образованную совершенно жестким твердым телом и жидкостью. Наряду с другими результатами получено подтверждение закона Бринкмана, а также четко определена область применения закона Дарси. Предложенная модель используется для изучения свободной конвекции в тонком пористом слое, нагреваемой снизу (неустойчивость Бенарда). Задача анализируется с помощью метода линейных возмущений: подчеркивается роль граничных условий и проницаемости на возникновение конвекции, особое внимание уделяется эффектам, обусловленным зависимостью вязкости от температуры.